DETERMINATION OF PERIODIC TRAJECTORIES IN Σ-Δ MODULATORS

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Summary: *In this paper we present an eigenvector approach for determination of k-periodic trajectories in various types of Σ-Δ modulators which can be modeled by a piece-wise linear map. The presented approach permits determination of the initial point (of k-periodic trajectories) via the values of the symbols of the admissible symbolic sequences and the eigenvalues and eigenvectors of the Jacobijan of the model.*

1. INTRODUCTION

The recent development of digital communications and signal processing technologies has increased the demands on precise and cheap AD converters. One such conversion scheme is Sigma-Delta modulation, which is particularly well suited for integrated circuit implementation due to the simplicity of its operation and the tolerance of eventual fabrication errors.

The main principle of operation of sigma–delta modulator is based on generation of coarse quantised (one bit) output signal by time sampling the input signal at frequencies much higher than its Niquist frequency. Despite the simplicity of this converter operation, the complete understanding of its operation is far from complete. This is mainly due to the nonlinear nature of the sigma-delta process, which results to highly complex behavior in even basic structures of the first-order (single-loop) modulators [1], [2].

In absence of a comprehensive theory of sigma-delta modulators, most researches have relied on standard linear analysis techniques ("first linearize, then analyze"). This approach has given very useful results but it was not sufficient for more profound understanding of their operation. In the last ten years, there are various approaches, which have been applied successfully to the ΣΔ modulators.

Among those methods, the dominant place has nonlinear dynamic theory in which, the main step toward the complete analysis of the system dynamics is determination of admissible k-periodic symbolic sequences and their corresponding periodic trajectories [3]. The solution of this problem could be very complex for large values of k, and any reduction of number of necessary arithmetic operations could enable creation of more effective algorithms for such purpose.

2. ΣΔ MODULATOR MODELS

Since the structure of the ΣΔ modulators is out of the scope of interest for this text, we shall concentrate only on the dynamical models of those systems.

One of the most widely exploited models of ΣΔ modulation is the non-ideal model of a double loop low-pass modulation. Its dynamic could be expressed by the set of difference equations:

$$\begin{aligned} x_1(n+1) &= p_2 x_1(n) + p_1 x_2(n) - u(n) - 2\text{sign}(x_1(n)) \\ x_2(n+1) &= p_1 x_2(n) - u(n) - \text{sign}(x_1(n)) \end{aligned} \quad (1)$$

where: $\mathbf{w}=[x_1,x_2]^T$, is the system state vector, u – is input signal, and p_1 and p_2 are the ΣΔ modulation parameters. Similarly, the dynamic of the band-pass Σ-Δ modulator can be described by

$$\begin{aligned} x(n+1) &= x(n+1) \\ x(n+2) &= -x(n)+2p_1 x(n+1) - 2p_1\text{sign}(x(n+1)+\text{sign}(x(n))+u(n) \end{aligned} \quad (2)$$

where: $\mathbf{w}(n)=[x(n),x(n+1)]$ is the state vector, $p_1=2\cos(\theta)$ and $\theta= 2\pi f_0/f_s$ is a parameter which determines the centre frequency f_0 of the pass-band filter used in the modulation (with sampling frequency $f_s >> f_0$).

In both cases, the dynamic of each model operating in autonomous mode (u(k) =0), can be described by piece-wise linear map [4]

$$\mathbf{w}(k+1) = \mathbf{A}\mathbf{w}(k) + \mathbf{b}s_k \quad \text{for} \quad \mathbf{w} \in D_i,\ s_k = \bar{s}_i \quad (3)$$

defined over domain R^2, where: **w** is 2x1 system vector state, **A** is lower left (or upper right) triangular 2x2 matrix, **b** is 2x1 vector, D_i, i=1,2,...,m are disjoint domains on which R^2 is partitioned ($R^2 = D_1 \cup D_2 \cup \ldots \cup D_m$; $D_i \cap D_j = \varnothing$, for i≠j) and $\bar{s}_i, i=1,2,..,m$ are real values which enable $\mathbf{Aw} + \mathbf{b}\bar{s}_i \in R^2$ for any $\mathbf{w} \in D_i$.

For model (1), sub-domains on which R^2 is partitioned are the upper and the lower half of the (x_1,x_2) plane with corresponding symbols $s_1=1$ and $s_2=-1$, while for the model (2), the domain R^2 is partitioned on four sub-domains that correspond to the four quadrants of (x(k+1),x(k+2)) plane with corresponding symbol values:

$s_1= 2p_1+1$; $s_2 = -2p_1+1$; $s_3 = -2p_1-1$ and $s_4 = 2p_1-1$.

By defining an alphabet of symbols $\hat{L} = \{\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_m\}$ with elements that correspond to the values of $\bar{s}_i, i=1,2,..,m$, one can define Σ as a set of all infinite symbolic sequences consisting of the symbols in $\hat{L}$. Hence, the mapping (3), starting from any initial point $\mathbf{w_0} = \mathbf{F}^0(w)$, $\mathbf{w_0} \in R^2$ generates symbolic sequence, $\tilde{s}$ ($\tilde{s} \in \Sigma$, $\tilde{s} = \tilde{s}_0\tilde{s}_1\tilde{s}_2....\tilde{s}_j..$) composed of symbols $\tilde{s}_j = \hat{s}_i$ if $w_j=\mathbf{F}^j(w) \in D_i$. Each of those symbolic sequences is called "admissible" and its symbols describe the order in which trajectories, starting from any initial condition, visit the various subregions of the phase space R^2. Therefore, it is obvious that any k-periodic trajectory generates k-periodic symbolic sequence $\tilde{s}$ ($\tilde{s} = \tilde{s}_0\tilde{s}_1\tilde{s}_2....\tilde{s}_{k-1}\tilde{s}_0\tilde{s}_1\tilde{s}_2....\tilde{s}_{k-1}...$), (periodic output bit-stream) which satisfies the conditions given by the following lemma:

Lemma 1: A point $\mathbf{w}_0$ is periodic point of period k, if and only if it satisfies

$$\mathbf{w}_0 = \mathbf{A}\mathbf{w}_{k-1} + \mathbf{b}s_{k-1}$$
$$\mathbf{w}_1 = \mathbf{A}\mathbf{w}_0 + \mathbf{b}s_0 \quad (4)$$
$$\mathbf{w}_{k-1} = \mathbf{A}\mathbf{w}_{k-2} + \mathbf{b}s_{k-2}$$

for some s_0, s_1,., s_{k-1} that corresponds to the symbols $\tilde{s}_0, \tilde{s}_1, .., \tilde{s}_{k-1}$, which are elements of $\hat{L}$ and $\mathbf{x}_j \in D_i$ if $s_j = \bar{s}_i$; j=0,1,...,k-1; i =1,2,..., m, such that $\mathbf{w}_j \neq \mathbf{w}_p$ for j≠p.

In other cases, the map exhibits, either fractal trajectories or dense trajectories in a form of ellipses.

3. DETERMINATION OF K-PERIODIC TRAJECTORIES

On basis of the **Lemma 1**, the conventional approach on determination of k-periodic trajectories and corresponding admissible k-periodic symbolic sequences includes two steps:

Step 1: Express the initial k-periodic point $\mathbf{w}_0$ as a function of **A, b** and s_0, s_1, ..., s_{k-1}

Step 2: For each possible k-periodic symbolic sequence $\tilde{s} = \tilde{s}_0\tilde{s}_1\tilde{s}_2....\tilde{s}_{k-1}\tilde{s}_0\tilde{s}_1...$; $\tilde{s}_j \in L$, j=0,1,...,k-1 with corresponding values s_0, s_1,..., s_{k-1}, determine the points $\mathbf{w}_j = F^j(\mathbf{w}_0)$, j=1,2,...,k-1 and check if they satisfy lema 1.

For any initial point $\mathbf{w}_0$, the k-tk iteration of the map can be determined by using

$$w(k) = \mathbf{A}^k w(0) + \sum_{j=0}^{k-1} \mathbf{A}^{k-1-j}\mathbf{b}s_j \quad (5)$$

Since the k-th iteration of any k-periodic point satisfies $\mathbf{w}_k = \mathbf{w}_0$, the initial condition can be expressed by

$$w(0) = (I - \mathbf{A}^k)^{-1} \sum_{j=0}^{k-1} \mathbf{A}^{k-1-j}\mathbf{b}s_j \quad (6)$$

only if $\det(I - \mathbf{A}^k) \neq 0$.

After this, one can proceed with step 2 of the conventional algorithm for determination of periodic trajectories. If the map consist alphabet of m symbols, then one should repeat steps 1 and 2, m^k times in order to find the admissible k-periodic symbolic sequences and their corresponding periodic trajectories.

4. EIGENVECTOR APROACH FOR DETERMINATION OF PERIODIC TRAJECTORIES

In case of non-ideal double loop low pass Σ-Δ modulator model, the Jacobian of the model (matrix **A**) has 2 two different eigenvalues, $\lambda_1 = p_1$ and $\lambda_2 = p_2$, with corresponding eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$ If we suppose that the modules of the both eigenvaules are non equal to 1, $|\lambda_1| \neq 1$ and $|\lambda_2| \neq 1$, then any initial condition $\mathbf{x}(0)$ and any constant vector **b** can be expressed as linear combination in a basis of eigenvectors of **A**, such as

$$w(0) = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 \text{ and } \quad \mathbf{b} = \beta_1\mathbf{e}_2 + \beta_2\mathbf{e}_2$$

By combining the last two equations with (5), we obtain

$$w(k) = \mathbf{A}^k \sum_{i=1}^{2} \alpha_i \mathbf{e}_i + \sum_{j=0}^{k-1} \mathbf{A}^{k-1-j} \left(\sum_{i=1}^{2} \beta_i \mathbf{e}_i \right) s_j \quad (7)$$

If we suppose that the trajectory of the system is k-periodic, $\mathbf{w}(k) = \mathbf{w}(0)$, from (5) we have

$$\sum_{i=1}^{2}\alpha_i\mathbf{e}_i = \mathbf{A}^k\sum_{i=1}^{2}\alpha_i\mathbf{e}_i + \sum_{j=0}^{k-1}\mathbf{A}^{k-1-j}\left(\sum_{i=1}^{2}\beta_i\mathbf{e}_i\right)s_j \tag{8}$$

$$\sum_{i=1}^{2}\alpha_i\mathbf{e}_i = \sum_{i=1}^{2}\alpha_i\mathbf{A}^k\mathbf{e}_i + \sum_{i=1}^{2}\beta_i\left(\sum_{j=0}^{k-1}\mathbf{A}^{k-1-j}\mathbf{e}_i s_j\right) \tag{9}$$

$$\sum_{i=1}^{2}\alpha_i\mathbf{e}_i = \sum_{i=1}^{2}\alpha_i\lambda_i^k\mathbf{e}_i + \sum_{i=1}^{2}\beta_i\left(\sum_{j=0}^{k-1}\lambda_i^{k-1-j}s_j\mathbf{e}_i\right) \tag{10}$$

Since the eigenvectors of **A** are the basis of R^2, the last expression can be decomposed on two independent equations

$$\alpha_i = \alpha_i\lambda_i^k + \beta_i\sum_{j=0}^{k-1}\lambda_i^{k-1-j}s_j \qquad i=1,2 \tag{11}$$

from which, we can uniquely determine the coefficients α_i whenever $1-\lambda_i^k \neq 0$

$$\alpha_i = \frac{\beta_i}{(1-\lambda_i^k)}\sum_{j=0}^{k-1}\lambda_i^{k-1-j}s_j \qquad i=1,2 \tag{12}$$

At last, by substitution of α_i in (5) we obtain the expression for the initial point, which is just candidate to be k - periodic for any k- periodic symbolic sequence $\tilde{s}$

$$\mathbf{w}(0)=\sum_{i=1}^{2}\left[\frac{\beta_i}{(1-\lambda_i^k)}\sum_{j=0}^{k-1}\lambda_i^{k-1-j}s_j\right]\mathbf{e}_i \tag{13}$$

The last expression has no any matrix multiplication and therefore is more convenient than (3) for developing faster computer algorithm for determination of admissible symbolic sequences and its corresponding periodic trajectories. It also considerably simplifies the periodic trajectory determination on any autonomous model of ΣΔ modulator, which is shown in the following section on the model (1).

The presented approach can not be applied on the model of the band-pass sigma delta modulator described with Eq (2) since it has Jacobian which has two equal eigenvalues, $\lambda_1=1$ and $\lambda_2=1$, that don't satisfy the condition $1-\lambda_i^k \neq 0; i=1,2$. In spite of that, the proposed approach, Eq (11), gives the following constrain

$$\sum_{j=0}^{k-1}\lambda_i^{k-1-j}s_j = \sum_{j=0}^{k-1}s_j = 0 \tag{14}$$

which reduces the number of searches for finding the admissible sequences.

5. EXAMPLE

We observe the map of the low-pass, double loop, sigma delta modulator where

$$A_1 = \begin{bmatrix} p_2 & p_1 \\ 0 & p_1 \end{bmatrix}; \quad b_1 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}; \quad L_1 = \begin{Bmatrix} s_1 = 1 \\ s_2 = -1 \end{Bmatrix}$$

Since **A** is upper left triangular matrix, one can easily determine p_2 and p_1 as eigenvalues of **A** and its corresponding eigenvectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix} 1 \\ -\frac{p_2 - p_1}{p_1} \end{bmatrix}.$$

Hence, the coordinates of the initial k-periodic point, $\mathbf{x}(0) = [x_1(0), x_2(0)]^T$, becomes

$$x_1(0) = -\frac{2p_2 - p_1}{(p_2 - p_1)(1 - p_2^k)} \sum_{j=0}^{k-1} p_2^{k-1-j} s_j + \frac{p_1}{(p_2 - p_1)(1 - p_1^k)} \sum_{j=0}^{k-1} p_1^{k-1-j} s_j$$

$$x_2(0) = -\frac{1}{(1 - p_1^k)} \sum_{j=0}^{k-1} p_1^{k-1-j} s_j$$

In case k=1 and k=2 we obtain the equations for fixed points and period 2 points, respectively, which are identical as in [1]

6. CONCLUSION

In this paper, we have presented an approach for finding admissible k-periodic sequences and its corresponding k-periodic trajectories in the ΣΔ modulator. The triangular form of the Jacobian matrix enable immediate determination of its eigenvalues and its eigenvectors, which are further used to obtain rather simplified expression for the initial k-periodic point as a function of the symbols of the admissible k-periodic sequence.

7. REFERENCES

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